ON MORI'S THEOREM FOR QUASICONFORMAL MAPS IN THE n-SPACE

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In memoriam: M.K. Vamanamurthy, 5 September 1934– 6 April 2009

Abstract. R. Fehlmann and M. Vuorinen proved in 1988 that Mori's constant $M(n,K)$ for K-quasiconformal maps of the unit ball in $\mathbf{R}^n$ onto itself keeping the origin fixed satisfies $M(n,K)\to 1$ when $K\to 1$. We give here an alternative proof of this fact, with a quantitative upper bound for the constant in terms of elementary functions. Our proof is based on a refinement of a method due to G.D. Anderson and M. K. Vamanamurthy. We also give an explicit version of the Schwarz lemma for quasiconformal self-maps of the unit disk. Some experimental results are provided to compare the various bounds for the Mori constant when $n=2$.

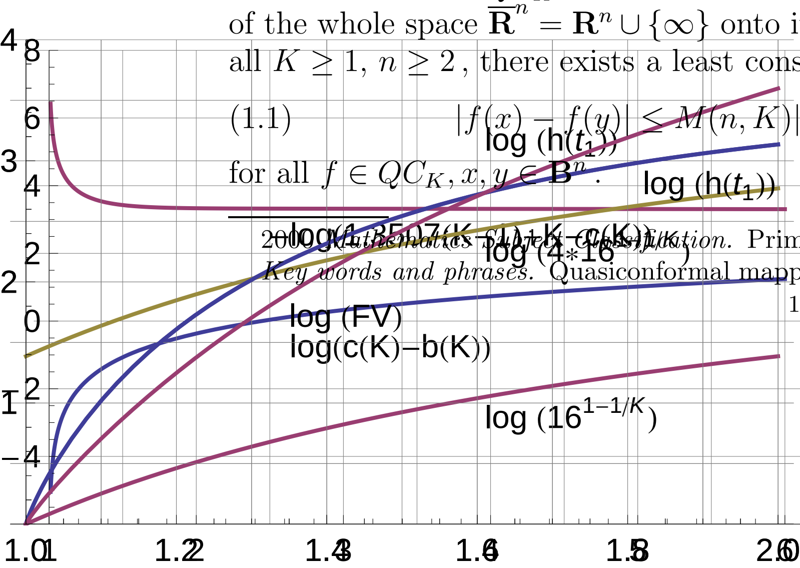
1. Introduction

Distortion theory of quasiconformal and quasiregular mappings in the Euclidean n-space $\mathbf{R}^n$ deals with estimates for the modulus of continuity and change of distances under these mappings. Some of the examples are the Hölder continuity, the quasiconformal counterpart of the Schwarz lemma, and Mori's theorem. The investigation of these topics started in the early 1950's for the case $n=2$ and ten years later for the case $n\geq 3$. Many authors have contributed to the distortion theory, for some historical remarks see [Vu1, 11.50].

As in [FV] we define Mori's constant $M(n,K)$ in the following way. Let QC_K, $K\geq 1$, stand for the family of all K-quasiconformal maps of the unit ball $\mathbf{B}^n$ onto itself keeping the origin pointwise fixed. Note that it is a well-known basic fact that an element in the set QC_K can be extended by reflection to a K-quasiconformal map of the whole space $\overline{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$ onto itself keeping the point ∞ fixed. Then for all $K\geq 1$, $n\geq 2$, there exists a least constant $M(n,K)\geq 1$ such that

$$|f(x)-f(y)|\leq M(n,K)|x-y|^{\alpha},\quad \alpha=K^{1/(1-n)}, \tag{1.1}$$

for all $f\in QC_K, x,y\in\mathbf{B}^n$.



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L. V. Ahlfors [A1] proved in 1954 that $M(2,K) \le 12^{K^2}$ and this property was refined by A. Mori [Mo] in 1956 to the effect that $M(2,K) \le 16$ and 16 cannot be replaced by a smaller constant independent of K. This result can also be found in [A2], [FM], and [LV]. On the other hand the trivial observation that 16 fails to be a sharp constant for $K = 1$ led to the following conjecture, which is still open in 2009.

1.2. **The Mori Conjecture.** $M(2,K) = 16^{1-1/K}$.

O. Lehto and K.I. Virtanen demonstrated in 1973 [LV, pp. 68] that $M(2,K) \ge 16^{1-1/K}$ (this lower bound was not given in the 1965 German edition of the book). It is natural to expect that for a fixed $n \ge 2$, $M(n,K) \to 1$ when $K \to 1$ and this convergence result with an explicit upper bound for $M(n,K)$ was proved by R. Fehlmann and M. Vuorinen [FV]. A counterpart of this result for the chordal metric was proved recently by P. Hästö in [H].

1.3. **Theorem.** [FV, Theorem 1.3] *Let f be a K-quasiconformal mapping of $\mathbf{B}^n$ onto $\mathbf{B}^n$, $n \ge 2$, $f(0) = 0$. Then*

$$|f(x) - f(y)| \le M(n,K)|x-y|^{\alpha} \tag{1.4}$$

for all $x, y \in \mathbf{B}^n$ where $\alpha = K^{1/(1-n)}$ and the constant $M(n,K)$ has the following three properties:

(1) *$M(n,K) \to 1$ as $K \to 1$, uniformly in n,*
(2) *$M(n,K)$ remains bounded for fixed K and varying n,*
(3) *$M(n,K)$ remains bounded for fixed n and varying K.*

For $n = 2$, the first majorants with the convergence property in 1.3(1) were proved only in the mid 1980s and for $n \ge 3$ in [FV]. In [FV] a survey of the various known bounds for $M(n,K)$ when $n \ge 2$ can be found – that survey reflects what was known at the time of publication of [FV]. Some earlier results on Hölder continuity had been proved in [G], [MRV], [R], [S]. Step by step the bound for Mori's constant was reduced during the past twenty years. As far as we know, the best upper bound known today for $n = 2$ is $M(2,K) \le 46^{1-1/K}$ due to S.-L. Qiu [Q] (1997). Refining the parallel work [FV], G.D. Anderson and M. K. Vamanamurthy proved the following theorem in [AV].

1.5. **Theorem.** *For $n \ge 2, K \ge 1$,*

$$M(n,K) \le 4\lambda_n^{2(1-\alpha)},$$

where $\alpha = K^{1/(1-n)}$ and $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$, is the Grötzsch ring constant [AN], [Vu1, p.89].

The first main result of this paper is Theorem 1.6 which improves on Theorem 1.5.

1.6. **Theorem.** *(1) For $n \ge 2, K \ge 1$, $M(n,K) \le T(n,K)$*

$$T(n,K) \le \inf\{h(t) : t \ge 1\}, \quad h(t) = (3 + \lambda_n^{\beta-1} t^{\beta}) t^{-\alpha} \lambda_n^{2(1-\alpha)}, \ t \ge 1, \tag{1.7}$$

where $\alpha = K^{1/(1-n)} = 1/\beta$, and λ_n is as in Theorem 1.5.

(2) There exists a number $K_1 > 1$ such that for all $K \in (1, K_1)$ the function h has a minimum at a point t_1 with $t_1 > 1$ and

(1.8)
$$T(n,K) \le h(t_1) = \left[\frac{3^{1-\alpha^2}(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}} \lambda_n^{\alpha-\alpha^2} + \lambda_n^{\beta-1} \left(\frac{(3\alpha)^{\alpha} \lambda_n^{\alpha-1}}{(\beta-\alpha)^{\alpha}} \right)^{\beta-\alpha} \right] \lambda_n^{2(1-\alpha)} .$$

Moreover, for $\beta \in (1,2)$ we have

$$h(t_1) \le 3^{1-\alpha^2} 2^{5(1-\alpha)} K^5 \left(\frac{3}{2} \sqrt[4]{\beta-\alpha} + \exp(\sqrt{\beta^2-1}) \right). \tag{1.9}$$

In particular, $h(t_1) \to 1$ when $K \to 1$.

The last statement shows that Theorem 1.6 is better than the result of Anderson and Vamanamurthy, Theorem 1.5, at least for values of K close to the critical value 1, because the constant of Theorem 1.5 satisfies $4\lambda_n^{2(1-\alpha)} \ge 4$.

The main method of our proof is to replace the argument of Anderson and Vamanamurthy by a more refined inequality from [Vu2] and to introduce an additional parameter (t in the above theorem) which will be chosen in an optimal way. The fact that this refined inequality is essentially sharp for values of t large enough, was recently proved by V. Heikkala and M. Vuorinen in [HV]. This gave us a hint that the inequality from [Vu2] might lead to an improvement of the results in [AV]. For the case $n = 2$ a numerical comparison of our bound (1.8) to Mori's conjectured bound, to the bound in Theorem 1.5 and to the bound in [FV] is presented in tabular and graphical form at the end of the paper.

We conclude this paper by discussing the Schwarz lemma for plane quasiconformal self-mappings of the unit disk, formulated in terms of the hyperbolic metric. The long history of this result is summarized in [Vu1, p.152, 11.50]. An up-to-date form of the Schwarz lemma was given in [Vu1, Theorem 11.2] and it will be stated for convenient reference also below as Theorem 4.4. A particular case, formula (4.6), was rediscovered by D.B.A. Epstein, A. Marden and V. Markovic [EMM, Thm 5.1].

We use the notations ch, th, arch and arth as in [Vu1], to denote the hyperbolic cosine, tangent and their inverse functions, resp. The second main result of this paper is an explicit form of the Schwarz lemma for quasiregular mappings, Theorem 1.10. We believe that in this simple form the result is new and perhaps of independent interest. The constant $c(K)$ below involves the transcendental function φ_K defined in Section 4.

1.10. **Theorem.** *If $f : \mathbf{B}^2 \to \mathbf{R}^2$ is a non-constant K-quasiregular mapping with $f\mathbf{B}^2 \subset \mathbf{B}^2$, and ρ is the hyperbolic metric of $\mathbf{B}^2$, then*

$$\rho(f(x), f(y)) \le c(K) \max\{\rho(x,y), \rho(x,y)^{1/K}\}$$

for all $x, y \in \mathbf{B}^2$ where $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ and

$$K \le u(K-1) + 1 \le \log(\mathrm{ch}(K \mathrm{arch}(e))) \le c(K) \le v(K-1) + K$$

with $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$ *and* $v = \log(2(1+\sqrt{1-1/e^2})) < 1.3507$. *In particular,* $c(1) = 1$.

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2. The main results

We shall follow here the standard notation and terminology for K-quasiconformal and K-quasiregular mappings in the Euclidean n-space $\mathbf{R}^n$, see e.g. [V], [Vu1], and we also recall some basic notation. For the modulus $M(\Gamma)$ of a curve family Γ and its basic properties see [V] and [Vu1].

Let D and D' be domains in $\overline{\mathbf{R}}^n$, $K \geq 1$, and let $f : D \to D'$ be a homeomorphism. Then f is K-quasiconformal if

$$M(\Gamma)/K \leq M(f\Gamma) \leq KM(\Gamma)$$

for every curve family Γ in D [V].

For subsets $E, F, D \subset \overline{\mathbf{R}}^n$ we denote by $\Delta(E, F; D)$ the family of all curves joining E and F in D. For brevity we write $\Delta(E, F) = \Delta(E, F; \mathbf{R}^n)$. A ring is a domain in $\mathbf{R}^n$, whose complement consists of two compact and connected sets. If these sets are E and F, then the ring is denoted by $R(E, F)$. The capacity of a ring $R(E, F)$ is

$$\mathrm{cap}R(E, F) = M(\Delta(E, F)).$$

The complementary components of the Grötzsch ring $R_{G,n}(s)$ are $\overline{\mathbf{B}}^n$ and $[se_1, \infty], s > 1$, while those of the Teichmüller ring $R_{T,n}(t)$ are $[-e_1, 0]$ and $[te_1, \infty], t > 0$. The conformal capacities of $R_{G,n}(s)$ and $R_{T,n}(t)$ are denoted by

$$\begin{cases} \gamma_n(s) = M(\Delta(\overline{\mathbf{B}}^n, [se_1, \infty])) \\ \tau_n(t) = M(\Delta([-e_1, 0], [te_1, \infty])) \end{cases}$$

respectively. Here $\gamma_n : (1, \infty) \to (0, \infty)$ and $\tau_n : (0, \infty) \to (0, \infty)$ are decreasing homeomorphisms and they satisfy the fundamental identity

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1), \quad t > 1\,, \tag{2.1}$$

see e.g. [Vu1, 5.53].

For $n \geq 2$ and $K > 0$, the distortion function $\varphi_{K,n} : [0,1] \to [0,1]$ is a homeomorphism. It is defined by

$$\varphi_{K,n}(t) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))}, \quad t \in (0,1), \tag{2.2}$$

and $\varphi_{K,n}(0) = 0$, $\varphi_{K,n}(1) = 1$. For $n \geq 2, K \geq 1$ and $0 \leq r \leq 1$

$$\varphi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^\alpha, \quad \alpha = K^{1/(1-n)}\,, \tag{2.3}$$

$$\varphi_{1/K,n}(r) \geq \lambda_n^{1-\beta} r^\beta, \quad \beta = K^{1/(n-1)}\,, \tag{2.4}$$

by [Vu1, Theorem 7.47] and where $\lambda_n \geq 4$ is as in Theorem 1.5.

2.5. **Lemma.** *Suppose that $f : \mathbf{B}^n \to \mathbf{B}^n$ is a K-quasiconformal mapping with $f\mathbf{B}^n = \mathbf{B}^n$, $f(0) = 0$, and let $h : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be the inversion $h(x) = x/|x|^2\,, h(\infty) = 0, h(0) = \infty$, and define $g : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ by $g(x) = f(x)$ for $x \in \mathbf{B}^n, g(x) = h(f(h(x)))$ for $x \in \mathbf{R}^n \setminus \overline{\mathbf{B}}^n$ and $g(x) = \lim_{z\to x} f(z)$ for $x \in \partial\mathbf{B}^n, g(\infty) = \infty$. Then g is a K-quasiconformal mapping, and we have for $x \in \mathbf{B}^n$*

$$\varphi_{1/K,n}(|x|) \leq |f(x)| \leq \varphi_{K,n}(|x|). \tag{2.6}$$

For $x \in \mathbf{R}^n \setminus \overline{\mathbf{B}}^n$

$$1/\varphi_{K,n}(1/|x|) \leq |g(x)| \leq 1/\varphi_{1/K,n}(1/|x|). \tag{2.7}$$

Proof. It is well-known that the above definition defines g as a K-quasiconformal homeomorphism. The formula (2.6) is well-known (see [AVV2, Theorem 4.2]) and (2.7) follows easily. $\square$

2.8. **Lemma.** [Vu1, Lemma 7.35] *Let $R = R(E,F)$ be a ring in $\overline{\mathbf{R}}^n$ and let $a, b \in E, c, d \in F$ be distinct points. Then*

$$capR = M(\Delta(E,F)) \geq \tau_n \left(\frac{|a-c||b-d|}{|a-b||c-d|} \right).$$

Equality holds if $b = t_1 e_1, a = t_2 e_1, c = t_3 e_1, d = t_4 e_1$ and $t_1 < t_2 < t_3 < t_4$.

We consider Teichmüller's extremal problem, which will be used to provide a key estimate in what follows. For $x \in \mathbf{R}^n \setminus \{0, e_1\}, n \geq 2$, define

$$p_n(x) = \inf_{E,F} M(\Delta(E,F))$$

where the infimum is taken over all the pairs of continua E and F in $\overline{\mathbf{R}}^n$ with $0, e_1 \in E$ and $x, \infty \in F$. Note that Lemma 2.8 gives the lower bound for $p_n(x)$ in Lemma 2.9.

2.9. **Lemma.** [Vu2, Theorem 1.5] *For $z \in \mathbf{R}^n, |z| > 1$, the following inequalities hold:*

$$\tau_n(|z|) = p_n(-|z|e_1) \leq p_n(z) \leq p_n(|z|e_1) = \tau_n(|z|-1)$$

where $p_n(z)$ is the Teichmüller function. Furthermore, for $z \in \mathbf{R}^n \setminus \{0, e_1\}$, there exists a circular arc E with $0, e_1 \in E$ and a ray F with $z, \infty \in F$ such that

$$p_n(z) \le \tau_n\left(\frac{|z|+|z-e_1|-1}{2}\right) = M(\Delta(E,F)) \le \tau_n(|z|-1) \tag{2.10}$$

with equality in the first inequality both for $z=-se_1, s>0$, and for $z=se_1, s>1$.

2.11**. Notation.** For $t>0, x,y\in \mathbf{B}^n$, we write

$$D(t,x,y) = |x+t\frac{y}{|y|}| \text{ if } y\neq 0, \quad D(t,x,0)=|x+e_1|\,.$$

By the triangle inequality we have

$$t-|x| \le D(t,x,y) \le t+|x|\,. \tag{2.12}$$

2.13. **Theorem.** *For $n\ge 2, K\ge 1$, let $f:\overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping, with $f\mathbf{B}^n \subset \mathbf{B}^n$, $f(0)=0$ and $f(\infty)=\infty$. Then for $t\ge 1$, $x,y\in \mathbf{B}^n\setminus\{0\}$, we have*

$$\begin{aligned} |f(x)-f(y)| &\le (3+\varphi_{1/K,n}(1/t)^{-1})\varphi^2_{K,n}\left(\left(\frac{2|x-y|}{s_1+|x-y|}\right)^{1/2}\right)\\ &\le (3+\lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s_1+|x-y|}\right)^{\alpha},\ \alpha=K^{1/(1-n)}=1/\beta, \end{aligned}$$

where $s_1=\max\{a,b\}, a=t+|x|+D(t,y,x), b=t+|y|+D(t,x,y)$.

Proof. Let Γ be the family $\Delta(E,F)$ and let E and F be connected sets as in Lemma 2.9 with $x,y\in E, z,\infty\in F$, where $z=-tx/|x|$ and $\Gamma^{'}=f(\Gamma)$. By Lemma 2.8 and (2.10), we have

$$\begin{aligned} &\tau_n\left(\frac{|f(z)-f(x)|}{|f(x)-f(y)|}\right) \le M(\Gamma^{'}) \le KM(\Gamma)\\ \le\ & K\tau_n(u-1)\,, \quad u=\frac{|x-z|+|z-y|-|x-y|+2|x-y|}{2|x-y|}\,. \end{aligned}$$

The basic identity (2.1) yields

$$\begin{aligned} \gamma_n\left(\left(\frac{|f(z)-f(y)|+|f(x)-f(y)|}{|f(x)-f(y)|}\right)^{1/2}\right) &\le K\gamma_n\left((u)^{1/2}\right) \\ &= K\gamma_n\left(\left(\frac{t+|x|+D(t,y,x)+|x-y|}{2|x-y|}\right)^{1/2}\right). \end{aligned} \tag{2.14}$$

Applying γ_n^{-1} to (2.14) we have

$$\frac{|f(z)-f(y)|+|f(x)-f(y)|}{|f(x)-f(y)|} \ge \left(\gamma_n^{-1}\left(K\gamma_n\left(\left(\frac{a+|x-y|}{2|x-y|}\right)^{1/2}\right)\right)\right)^2 = v.$$

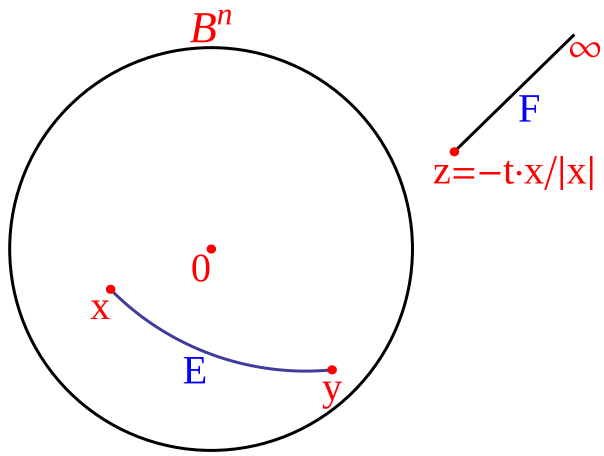

FIGURE 1. Geometrical meaning of the proof of Theorem 2.13.

Because $f\mathbf{B}^n \subset \mathbf{B}^n$, by (2.6) and (2.4) we know that

$$|f(z)-f(y)|+|f(x)-f(y)| \le 3+\varphi_{1/K,n}(1/t)^{-1} \le 3+\lambda_n^{(\beta-1)}t^\beta,$$

$$\frac{|f(x)-f(y)|}{3+\varphi_{1/K,n}(1/t)^{-1}} \le \frac{|f(x)-f(y)|}{|f(z)-f(y)|+|f(x)-f(y)|} \le 1/v, \tag{2.15}$$

also

$$\begin{aligned}|f(x)-f(y)| &\le (3+\varphi_{1/K,n}(1/t)^{-1})\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{a+|x-y|}\right)^{1/2}\right)\\ &\le (3+\lambda_n^{(\beta-1)}t^\beta)\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{a+|x-y|}\right)^{\alpha}\end{aligned}$$

by inequalities (2.2) and (2.3). Exchanging the roles of x and y we see that

$$\begin{aligned}|f(x)-f(y)| &\le (3+\varphi_{1/K,n}(1/t)^{-1})\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{s_1+|x-y|}\right)^{1/2}\right)\\ &\le (3+\lambda_n^{(\beta-1)}t^\beta)\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s_1+|x-y|}\right)^{\alpha}.\end{aligned}$$

$\square$

Setting $t=1$, we get the following corollary.

2.16. **Corollary.** *For $n \ge 2, K \ge 1$, let $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping, with $f\mathbf{B}^n \subset \mathbf{B}^n$, $f(0)=0$ and $f(\infty)=\infty$. Then for all $x, y \in \mathbf{B}^n \setminus \{0\}$,*

$$|f(x)-f(y)| \le 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s+|x-y|}\right)^{\alpha},$$

where $\alpha = K^{1/(1-n)}$ and $s = \max\{a,b\}, a = 1+|x|+D(1,y,x), b = 1+|y|+D(1,x,y)$.

Proof. The proof is similar to the above proof except that here we consider the particular case $t=1$. Because $f\mathbf{B}^n \subset \mathbf{B}^n$, we know that $|f(z)-f(y)|+|f(x)-f(y)| \le 4$,

$$\begin{aligned}\frac{|f(x)-f(y)|}{4} &\le \frac{|f(x)-f(y)|}{|f(z)-f(y)|+|f(x)-f(y)|}\\ &\le \frac{1}{\left(\gamma_n^{-1}\left(K\gamma_n\left(\left(\dfrac{a+|x-y|}{2|x-y|}\right)^{1/2}\right)\right)\right)^2},\end{aligned}$$

or

$$\begin{aligned}|f(x)-f(y)| &\le 4\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{a+|x-y|}\right)^{1/2}\right)\\ &\le 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{a+|x-y|}\right)^{\alpha}\end{aligned}$$

by inequalities (2.2) and (2.3). Exchanging the roles of x and y we get

$$|f(x)-f(y)| \le 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{\max\{a,b\}+|x-y|}\right)^{\alpha}.$$

□

2.17. **Corollary.** *For $n\ge 2, K\ge 1, t\ge 1$, let f be as in Theorem 2.13. Then*

$$|f(x)-f(y)| \le (3+\lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{2t+||x|-|y||+|x-y|}\right)^{\alpha}, \tag{2.18}$$

for all $x,y\in\mathbf{B}^n$,

$$|f(x)-f(y)| \le (3+\lambda_n^{\beta-1}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{|x-y|}{\max\{t+|x|,t+|y|\}}\right)^{\alpha}, \tag{2.19}$$

for all $x,y\in\mathbf{B}^n$, and

$$|f(x)-f(y)| \le (3+\lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{|x-y|}{t+|x|+(|x-y|)/2}\right)^{\alpha}, \tag{2.20}$$

if $D(t,y,x)>t+|x|, x,y\in\mathbf{B}^n$.

Proof. Inequality (2.18) follows because by (2.11) $D(t,y,x)>t-|y|$ and $D(t,x,y)>t-|x|$ for $x,y\in\mathbf{B}^n$, and hence, in the notation of Theorem 2.13,

$$s_1 \ge \max\{2t+|x|-|y|, 2t+|y|-|x|\} = 2t+||x|-|y||.$$

It is also clear that $D(t,y,x)\ge t+|x|-|x-y|$, and this implies that

$$s_1 \ge \max\{2(t+|x|)-|x-y|, 2(t+|y|)-|x-y|\} = 2\max\{t+|x|,t+|y|\}-|x-y|$$

and hence the inequality (2.19) follows. In the case of (2.20) we have $D(t,y,x) > t+|x|$ and see that, in the notation of Corollary 2.16, $s > 2(t+|x|)$ and (2.20) holds. $\square$

2.21. **Corollary.** *For $n \ge 2, K \ge 1$, let f be as in Theorem 2.13. Then*

$$|f(x)-f(y)| \le 4\lambda_n^{2(1-\alpha)} \left(\frac{2|x-y|}{2+||x|-|y||+|x-y|} \right)^{\alpha}, \tag{2.22}$$

for all $x, y \in \mathbf{B}^n \setminus \{0\}$.

2.23. **Remark.** (1) In several of the above results we have supposed that $x, y \in \mathbf{B}^n \setminus \{0\}$. If one of the points x, y were equal to 0, then we would have a better result from the Schwarz lemma estimate (4.7).

(2) Corollary 2.21 is an improvement of the Anderson-Vamanamurthy theorem 1.5.

3. Comparison with earlier bounds

3.1**. Proof of Theorem 1.6.** (1) The inequality (1.7) follows easily from the inequality (2.19).

(2) We see that the function h has a local minimum at $t_1 = (3\alpha)^{\alpha}\lambda_n^{\alpha-1}(\beta-\alpha)^{-\alpha}$. If $t_1 \ge 1$, then the inequality (2.19) yields the desired conclusion. The upper bound for $T(n,K)$ follows by substituting the argument t_1 in the expression of h.

We next show that the value $K_1 = 4/3$ will do. Fix $K \in (1, K_1)$. Then $\alpha = K^{1/(1-n)} \ge 3/4$ and $\alpha/(1-\alpha^2) > 1$.

Because $\lambda_n^{\alpha-1} \ge 2^{1/K-1}K^{-1}$ by [Vu1, Lemma 7.50(1)], with $d = (6/K)^{1/K}/2K$ we have

$$\begin{aligned} t_1 = (3\alpha)^{\alpha}\lambda_n^{\alpha-1}(\beta-\alpha)^{-\alpha} &\ge (3/K)^{1/K} 2^{1/K-1} K^{-1} \left(\frac{\alpha}{1-\alpha^2} \right)^{\alpha} \\ &= d \left(\frac{\alpha}{1-\alpha^2} \right)^{\alpha} \ge d \left(\frac{\alpha}{1-\alpha^2} \right)^{3/4} \\ &= \left(2r(K) \frac{\alpha}{1-\alpha^2} \right)^{3/4} \quad ; r(K) = d^{4/3}/2\,. \end{aligned}$$

It suffices to observe that $t_1 > 1$ certainly holds if $2r(K)(\frac{\alpha}{1-\alpha^2}) > 1$ which holds for $\alpha > 1/(r(4/3) + \sqrt{1+r(4/3)^2}) = 0.53...$, in particular, $t_1 > 1$ holds in the present case $\alpha > 3/4$.

For the proof of (1.9) we give the following inequalites

$$\lambda_n^{\alpha-\alpha^2} \le 2^{\alpha(1-\alpha)} K^{\alpha} \le 2^{1-\alpha} K^{\alpha}, \quad K \ge 1\,, \tag{3.2}$$

$$\lambda_n^{\beta-\alpha} = \lambda_n^{\beta+1-1-\alpha} = \lambda_n^{\beta(1-\alpha)+1-\alpha} = \lambda_n^{(\beta+1)(1-\alpha)} \le (2^{1-\alpha}K)^3, \quad \beta \in (1,2)\,, \tag{3.3}$$

see [Vu1, Lemma 7.50(1)]. The formula (1.8) for $h(t_1)$ has two terms. We estimate separately each term as follows

$$\begin{aligned}
\frac{3^{1-\alpha^2}(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}\lambda_n^{\alpha-\alpha^2}\lambda_n^{2(1-\alpha)} &\leq \frac{3^{(1-\alpha)(1+\alpha)}2^{\alpha(1-\alpha)}2^{2(1-\alpha)}K^2(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}K^{\alpha}\\
&\leq \frac{(9\cdot 2\cdot 4)^{1-\alpha}K^2(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}K^{\alpha}\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2K^{\alpha}\exp(-\alpha^2\log\alpha)\\
&\leq 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2K^{\alpha}\exp(-\alpha\log\alpha)\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp((\log K-\log\alpha)\alpha)\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp\left(\left(1+\frac{1}{n-1}\log K\right)\alpha\right)\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp\left(\frac{n}{n-1}\alpha\log K\right)\\
&\leq 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp(2\log K)\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^4
\end{aligned}$$

by inequality (3.2),

$$\begin{aligned}
\lambda_n^{2(1-\alpha)}\lambda_n^{\beta-1}\left(\frac{(3\alpha)^{\alpha}\lambda_n^{\alpha-1}}{(\beta-\alpha)^{\alpha}}\right)^{\beta-\alpha} &= \lambda_n^{2(1-\alpha)}\lambda_n^{\beta-1}\left((3\alpha)^{\alpha}\lambda_n^{\alpha-1}\right)^{\beta-\alpha}(\beta-\alpha)^{-\alpha(\beta-\alpha)}\\
&\leq (2^{1-\alpha}K)^2\lambda_n^{\beta-\alpha}\left((3\alpha)^{\alpha}\lambda_n^{\alpha-1}\right)^{\beta-\alpha}\left(\frac{\beta^2-1}{\beta}\right)^{-\alpha((\beta^2-1)/\beta)}\\
&\leq (2^{1-\alpha}K)^2\left(3^{\alpha}\lambda_n\right)^{\beta-\alpha}\beta^{\alpha^2}(\beta^2-1)^{-\alpha^2(\beta^2-1)}\\
&\leq (2^{1-\alpha}K)^2 3^{\alpha(\beta-\alpha)}\lambda_n^{(\beta+1)(1-\alpha)}\exp\left(\frac{2\alpha^2}{e}\sqrt{\beta^2-1}\right)\\
&\leq 3^{1-\alpha^2}(2^{1-\alpha}K)^2(2^{1-\alpha}K)^{(\beta+1)}\exp\left(\frac{2\alpha^2}{e}\sqrt{\beta^2-1}\right)\\
&\leq 3^{1-\alpha^2}(2^{1-\alpha}K)^5\exp(\sqrt{\beta^2-1}),
\end{aligned}$$

here we assume that $\beta\in(1,2)$ which implies that $\alpha\in(1/2,1)$. Also the inequalities $(K-1)^{-(K-1)}\leq\exp((2/e)\sqrt{K-1})$ and (3.3) were used, and we get

$$h(t_1)\leq\left[72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^4+3^{\beta-\alpha}(2^{1-\alpha}K)^5\exp(\sqrt{\beta^2-1})\right]. \tag{3.4}$$

Because $(\beta-\alpha)\in(0,\frac{3}{2})$ this implies that $\frac{2}{3}(\beta-\alpha)\in(0,1)$ and $\alpha^2\in(\frac{1}{4},1)$ and further $(\frac{2}{3}(\beta-\alpha))^{\alpha^2}\leq(\frac{2}{3}(\beta-\alpha))^{1/4}$, and finally

$$(\beta-\alpha)^{\alpha^2}\leq(2/3)^{-\alpha^2}(\frac{2}{3}(\beta-\alpha))^{1/4}\leq(3/2)^{3/4}\sqrt[4]{\beta-\alpha}$$

$$= (3/2)^{3/4}\sqrt[4]{\beta-\alpha} < (3/2)\sqrt[4]{\beta-\alpha}\,.$$

Next we prove that

$$72^{1-\alpha} \le 3^{1-\alpha^2} 2^{5(1-\alpha)} K\,. \tag{3.5}$$

This inequality is equivalent to

$$2^{2(\alpha-1)} 3^{(1-\alpha)^2} \le K \iff -(1-\alpha)\log 4 + (1-\alpha)^2 \log 3 \le \log K\,.$$

This last inequality holds because the left hand side is negative. Now from (3.4) and (3.5) we get the desired inequality (1.9). □

3.6. **Graphical and numerical comparision of various bounds.** The above bounds involve the Grötzsch ring constant λ_n, which is known only for $n=2, \lambda_2 = 4$. Therefore only for $n=2$ we can compute the values of the bounds. Solving numerically the equation $4\cdot 16^{1-1/K} = h(t_1)$ for K we obtain $K = 1.3089$. We give numerical and graphical comparison of the various bounds for the Mori constant.

Tabulation of the various upper bounds for Mori's constant when $n=2$ and $\lambda_2 = 4$ as a function of K: (a) Mori's conjectured bound $16^{1-1/K}$, (b) the Anderson-Vamanamurthy bound $4\cdot 16^{1-1/K}$, (c) the bound from (1.8). For $K \in (1, 1.3089)$ the upper bound in (1.8) is better than the Anderson-Vamanamurthy bound and for $K > 1.5946$ the upper bound in (1.8) is better than the bound of Fehlmann and Vuorinen. Numerical values of the [FV] bound given in the table were computed with the help of the algorithm for $\varphi_{K,2}(r)$ attached with [AVV1, p. 92, 439].

K	$\log(16^{1-1/K})$	$\log(4\cdot 16^{1-1/K})$	$\log(FV)$	$\log(h(t_1))$
1.1	0.2521	1.6384	0.7051	1.0188
1.2	0.4621	1.8484	1.2485	1.6058
1.3	0.6398	2.0261	1.7046	2.0107
1.4	0.7922	2.1785	2.0913	2.3061
1.5	0.9242	2.3105	2.4221	2.5296
1.6	1.0397	2.4260	2.7094	2.7031
1.7	1.1417	2.5280	2.9633	2.8409
1.8	1.2323	2.6186	3.1921	2.9521
1.9	1.3133	2.6996	3.4020	3.0433
2.0	1.3863	2.7726	3.5979	3.1192

For graphing and tabulation purposes we use the logarithmic scale. Note that the upper bound for $M(2,K)$ given in [FV, Theorem 2.29] also has the desirable property that it converges to 1 when $K \to 1$, see Figure 2.

3.7. **Comparison of estimates for the Hölder quotient.** For a K-quasiconformal mapping $f : \mathbf{B}^n \to f\mathbf{B}^n = \mathbf{B}^n$, we call the expression

$$HQ(f) = \sup\{|f(x)-f(y)|/|x-y|^\alpha :\ x,y\in\mathbf{B}^n, f(0)=0\ x\neq y\},$$

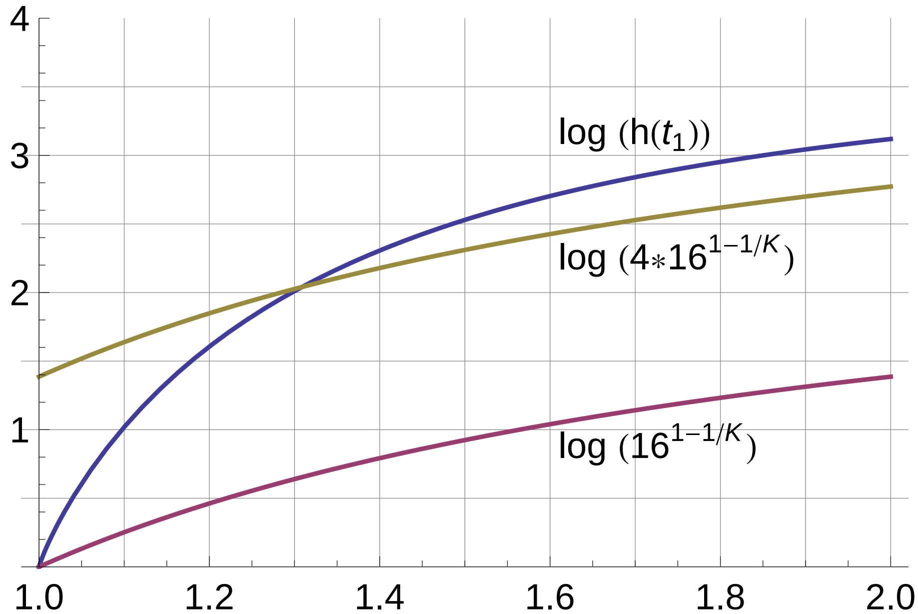

FIGURE 2. Graphical illustration of the various upper bounds for Mori's constant when $n = 2$ and $\lambda_2 = 4$ as a function of K: (a) Mori's conjectured bound $16^{1-1/K}$, (b) the Anderson-Vamanamurthy bound $4 \cdot 16^{1-1/K}$, (c) the bound from (1.8). For $K \in (1, 1.3089)$ the upper bound in (1.8) is better than the Anderson-Vamanamurthy bound.

the Hölder coefficient of f. Clearly $HQ(f) \le M(n,K)$. Theorem 2.13 yields, after dividing the both sides of the inequality in 2.13 by $|x-y|^{\alpha}$, the upper bound $HQ(f) \le HQ(K)$ for the Hölder quotient with

$$HQ(K) = \sup\{\inf\{U(t,x,y) : \ t \ge 1\} : \ x, y \in \mathbf{B}^n\}\,, \tag{3.8}$$

$$U(t,x,y) = (3 + \varphi_{1/K,n}(1/t)^{-1})\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{1/2}\right)\frac{1}{|x-y|^{\alpha}}\,.$$

For $n = 2$ we compare $HQ(K)$ to several other bounds (a) Mori's conjectured bound, (b) the FV bound, (c) the AV bound and give the results as a table and Figure 3. Because the supremum and infimum in (3.8) cannot be explicitly found we use numerical methods that come with Mathematica software. For the numerical tests we used for the supremum a sample of $100,000$ random points of the unit disk.

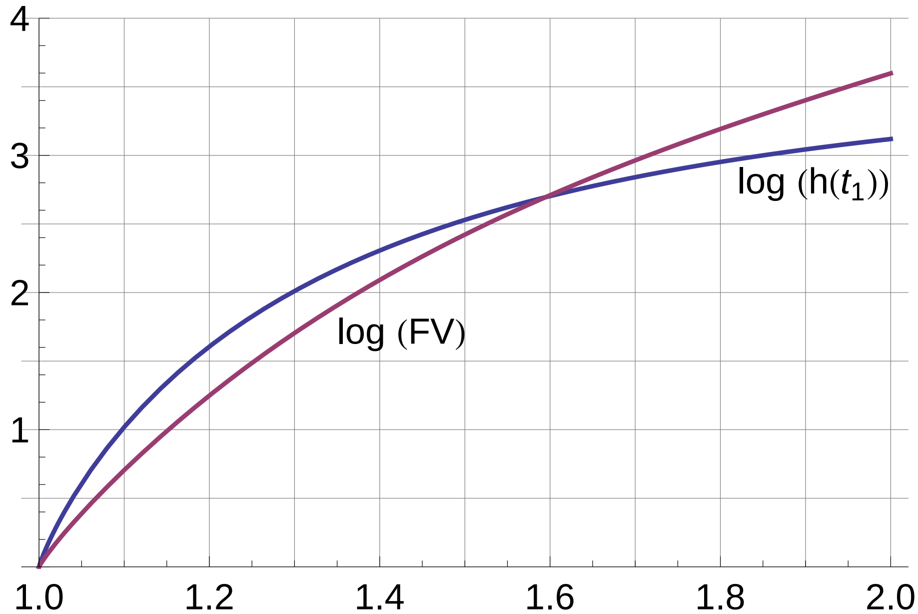

FIGURE 3. Graphical comparison of various bounds when $n = 2$ and $\lambda_2 = 4$, as a function of K: (a) the bound from (1.8), (b) the Fehlmann and Vuorinen bound [FV]

$$M(2,K) \le \left(1 + \varphi_{K,2}\left(\frac{K^2-1}{K^2+1}\right)\right) 2^{2K-3/K} \frac{(K^2+1)^{(K+1/K)/2}}{(K^2-1)^{(K-1/K)/2}}.$$

For $K > 1.5946$ the upper bound in (1.8) is better than the Fehlmann-Vuorinen bound.

K	$\log(16^{1-1/K})$	$\log(4 \cdot 16^{1-1/K})$	$\log(FV)$	$\log(HQ(K))$
1.1	0.2521	1.6384	0.7051	1.0171
1.2	0.4621	1.8484	1.2485	1.5940
1.3	0.6398	2.0261	1.7046	1.9712
1.4	0.7922	2.1785	2.0913	2.1668
1.5	0.9242	2.3105	2.4221	2.2928
1.6	1.0397	2.4260	2.7094	2.4003
1.7	1.1417	2.5280	2.9633	2.4922
1.8	1.2323	2.6186	3.1921	2.5706
1.9	1.3133	2.6996	3.4020	2.6371
2.0	1.3863	2.7726	3.5979	2.6934

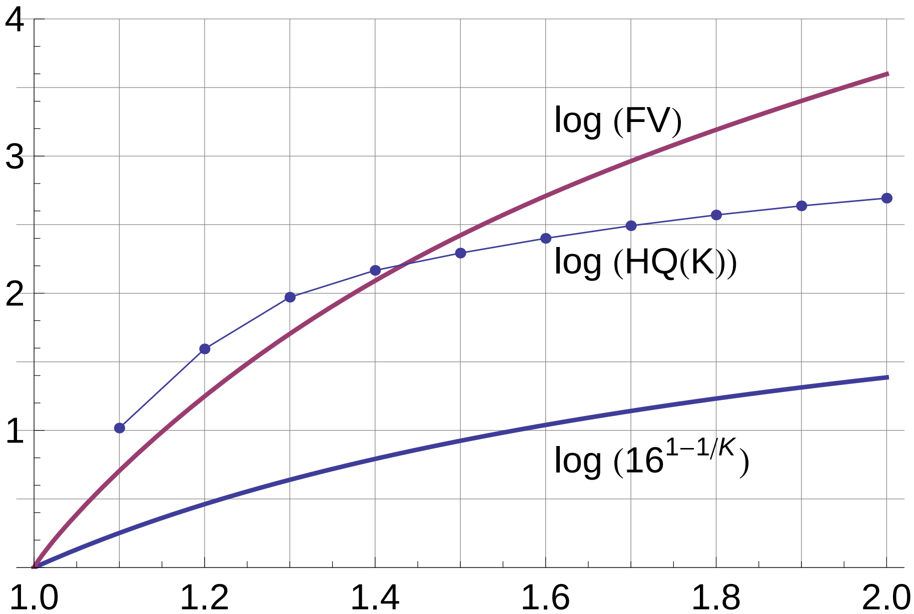

FIGURE 4. Graphical comparison of various bounds when $n = 2$ and $\lambda_2 = 4$, as a function of K: (a) the bound from (3.8), (b) the Fehlmann and Vuorinen bound [FV]

$$M(2,K) \le \left(1+\varphi_{K,2}\left(\frac{K^2-1}{K^2+1}\right)\right) 2^{2K-3/K}\frac{(K^2+1)^{(K+1/K)/2}}{(K^2-1)^{(K-1/K)/2}},$$

(c) the bound of the Mori conjecture. Note that the bound (3.8), based on a simulation with $100,000$ random points, gives the best estimate in the cases considered in the picture.

4. An explicit form of the Schwarz lemma

Recall that the hyperbolic metric $\rho(x,y), x,y \in \mathbf{B}^n$, of the unit ball is given by (cf. [KL], [Vu1])

$$\text{th}^2\frac{\rho(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2+t^2}, \quad t^2 = (1-|x|^2)(1-|y|^2)\,. \tag{4.1}$$

Next, we consider a decreasing homeomorphism $\mu : (0,1) \longrightarrow (0,\infty)$ defined by

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \tag{4.2}$$

where $\mathcal{K}(r)$ is Legendre's complete elliptic integral of the first kind and $r' = \sqrt{1-r^2}$, for all $r \in (0,1)$.

The Hersch-Pfluger distortion function is an increasing homeomorphism $\varphi_K : (0,1) \longrightarrow (0,1)$ defined by setting

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K)\,,\ r \in (0,1),\ K > 0. \tag{4.3}$$

Note that with the notation of Section 2, $\gamma_2(1/r) = 2\pi/\mu(r)$ and $\varphi_K(r) = \varphi_{K,2}(r)$ for $r \in (0,1)$.

4.4. **Theorem.** [Vu1, 11.2] *Let $f : \mathbf{B}^n \to \mathbf{R}^n$ be a nonconstant K-quasiregular mapping with $f\mathbf{B}^n \subset \mathbf{B}^n$ and let $\alpha = K^{1/(1-n)}$. Then*

$$\text{th}\frac{\rho(f(x),f(y))}{2} \le \varphi_{K,n}(\text{th}\frac{\rho(x,y)}{2}) \le \lambda_n^{1-\alpha}\left(\text{th}\frac{\rho(x,y)}{2}\right)^{\alpha}, \tag{4.5}$$

$$\rho(f(x),f(y)) \le K(\rho(x,y) + \log 4), \tag{4.6}$$

for all $x, y \in \mathbf{B}^n$, where λ_n is the same constant as in (1.5). *If $f(0) = 0$, then*

$$|f(x)| \le \lambda_n^{1-\alpha}|x|^{\alpha}, \tag{4.7}$$

for all $x \in \mathbf{B}^n$.

In the case of quasiconformal mappings with $n = 2$ formulas (4.5) and (4.7) also occur in [LV, p. 65] and formula (4.6) was rediscovered in [EMM, Theorem 5.1]. Comparing Theorem 4.4 to Theorem 1.10 we see that for $n = 2$ the expression $K(\rho(x,y) + \log 4)$ may be replaced with $c(K)\max\{\rho(x,y), \rho(x,y)^{1/K}\}$, which tends to 0 when $x \to y$ and to $\rho(x,y)$ when $K \to 1$, as expected.

4.8. **Lemma.** *For $K > 1$ the function*

$$t \mapsto \frac{2\text{arth}(\varphi_K(\text{th}\frac{t}{2}))}{\max\{t, t^{1/K}\}},$$

is monotone increasing on $(0,1)$ and decreasing on $(1,\infty)$.

Proof. (1) Fix $K > 1$ and consider

$$f(t) = \frac{2\text{arth}(\varphi_K(\text{th}\frac{t}{2}))}{t}, \quad t > 0.$$

Let $r = \text{th}\frac{t}{2}$. Then $t/2 = \text{arth}r$, and t is an increasing function of r for $0 < r < 1$. Then

$$f(t) = \frac{2\text{arth}(\varphi_K(\text{th}\frac{t}{2}))}{t/2} = \frac{\text{arth}(\varphi_K(r))}{\text{arth}r} = F(r).$$

Then by [AVV1, Theorem 10.9(3)], $F(r)$ is strictly decreasing from $(0,1)$ onto (K,∞). Hence $f(t)$ is strictly decreasing from $(0,\infty)$ onto (K,∞).

(2) Next consider

$$g(t) = \frac{2\text{arth}(\varphi_K(\text{th}\frac{t}{2}))}{t^{1/K}},$$

and let $r = \text{th}\frac{t}{2}$. Then $t = 2\text{arth}r$ and

$$g(t) = \frac{2\text{arth}s}{2^{1/K}(\text{arth}r)^{1/K}} = \frac{2^{1-1/K}\text{arth}s}{(\text{arth}r)^{1/K}},$$

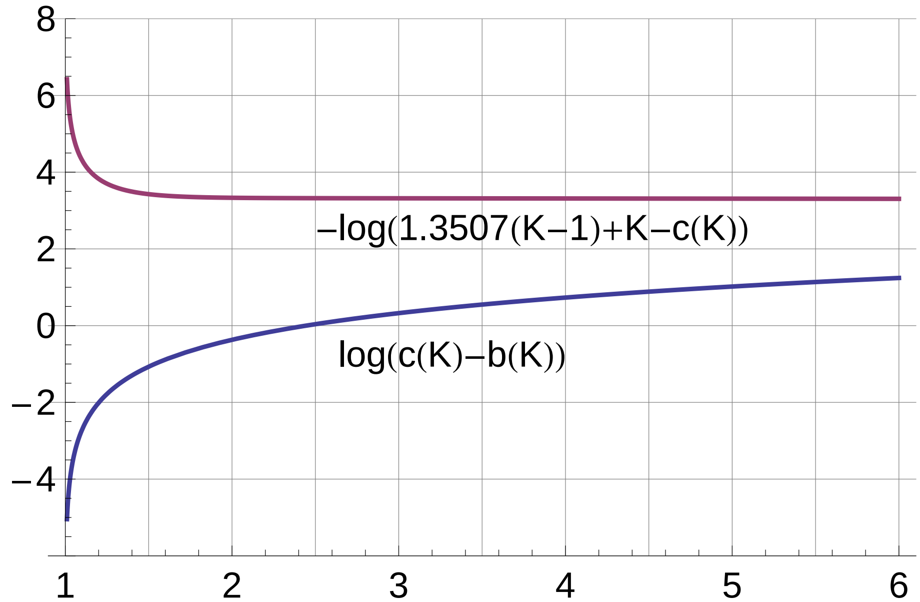

Figure 5. Graphical comparison of lower and upper bounds for $c(K)$ with $b(K) = \log(\mathrm{ch}(K\mathrm{arch}(e)))$.

where $s = \varphi_K(r)$. We next apply [AVV1, Theorem 1.25]. We know $\frac{d}{dr}(\mathrm{arth} r) = 1/(1-r^2)$.

Writing $r' = \sqrt{1-r^2}, s' = \sqrt{1-s^2}$, we obtain the quotient of the derivatives

$$\frac{2^{1-1/K}(1/(1-s^2))\frac{ds}{dr}}{\frac{1}{K}(\mathrm{arth} r)^{1/K-1}(1/(1-r^2))} = 2^{1-1/K}\,K\,(\mathrm{arth} r)^{1-1/K}\frac{r'^2}{s'^2}\frac{1}{K}\frac{ss'^2\,\mathcal{K}(s)^2}{rr'^2\,\mathcal{K}(r)^2}$$

$$= 2^{1-1/K}(\mathrm{arth} r)^{1-1/K}\frac{s\,\mathcal{K}(s)^2}{r\,\mathcal{K}(r)^2}$$

by [AVV1, appendix E(23)]. By [AVV1, Lemma 10.7(3)], $\frac{\mathcal{K}(s)^2}{\mathcal{K}(r)^2}$ is increasing, since $K > 1$, $(\mathrm{arth} r)^{1/K-1}$ is increasing. Finally, s/r is increasing by [AVV1, Theorem 1.25] and E(23). So $g(t)$ is increasing in t on $(0, \infty)$.

(3) Fix $K > 1$. Clearly

$$\max\{t, t^{1/K}\} = \begin{cases} t^{1/K} & \text{for} \quad 0 \le t \le 1 \\ t & \text{for} \quad 1 \le t < \infty. \end{cases}$$

Thus

$$h(t) = \frac{2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{\max\{t, t^{1/K}\}},$$

increases on $(0, 1)$ and decreases on $(1, \infty)$. $\square$

4.9**. Proof of Theorem 1.10.** The maximum value of the function considered in Lemma 4.8 is $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$. The inequality now follows from Lemma 4.8. $\square$

4.10**. Bounds for the constant** $c(K)$**.** In order to give upper and lower bounds for $c(K)$, we observe that the identity [AVV1, Theorem 10.5(2)] yields the following formula

$$c(K) = 2\mathrm{arth}\left(\varphi_K\left(\frac{1-1/e}{1+1/e}\right)\right) = 2\mathrm{arth}\left(\frac{1-\varphi_{1/K}(1/e)}{1+\varphi_{1/K}(1/e)}\right).$$

A simplification leads to

$$c(K) = -\log \varphi_{1/K}(1/e)\,.$$

Next, from the inequality $\varphi_{1/K}(r) \geq 2^{1-K}(1+r')^{1-K}r^K$ for $K \geq 1, r \in (0,1)$ (cf. [AVV1, Corollary 8.74(2)]) we get with $v = \log(2(1+\sqrt{1-1/e^2})) < 1.3507$

$$\begin{aligned} c(K) &= -\log \varphi_{1/K}(1/e) \leq -\log\left(2^{1-K}(1+\sqrt{1-1/e^2})^{1-K}e^{-K}\right) \\ &= v(K-1)+K < 1.3507(K-1)+K. \end{aligned}$$

In order to estimate the constant $c(K)$ from below we need an upper bound for $\varphi_{1/K,2}(r),\ K>1$, from above. For this purpose we prove the following lemma.

4.11. **Lemma.** *For every integer $n \geq 2$ and each $K > 1,\ r \in (0,1)$, there exists K-quasiconformal maps $g : \mathbf{B}^n \to \mathbf{B}^n$ and $h : \mathbf{B}^n \to \mathbf{B}^n$ with*

(a) $\quad g(0) = 0,\ g\mathbf{B}^n = \mathbf{B}^n,\ h(0) = 0,\ h\mathbf{B}^n = \mathbf{B}^n$

(b) $\quad g(re_1) = \dfrac{2r^\alpha}{(1+r')^\alpha + (1-r')^\alpha},\ h(re_1) = \dfrac{2r^\beta}{(1+r')^\beta + (1-r')^\beta}$

where $r' = \sqrt{1-r^2}$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

In particular, for $n = 2$ and $K > 1,\ r \in (0,1)$

(c) $\quad \varphi_{1/K}(r) \leq \dfrac{2r^K}{(1+r')^K + (1-r')^K}\ ;\ \ \varphi_K(r) \geq \dfrac{2r^{1/K}}{(1+r')^{1/K} + (1-r')^{1/K}}.$

Proof. Fix $r \in (0,1)$. Let $T_a : \mathbf{B}^n \to \mathbf{B}^n$ be a Möbius automorphism with $T_a(a) = 0$ and $T_a(\mathbf{B}^n) = \mathbf{B}^n$. Choose $s \in (0,r)$ such that $T_{se_1}(0) = -T_{se_1}(re_1)$. Then $\rho(0,re_1) = 2\rho(0,se_1)$ [Vu1, (2.17)], or equivalently, $(1+r)/(1-r) = ((1+s)/(1-s))^2$ and hence $s = r/(1+r')$. Consider the K-quasiconformal mapping $f : \mathbf{B}^n \to \mathbf{B}^n$, $f(x) = |x|^{\alpha-1}x,\ \alpha = K^{1/(1-n)}$. Then $f(\pm se_1) = \pm s^\alpha e_1$. The mapping $g = T_{-s^\alpha e_1} \circ f \circ T_{se_1} : \mathbf{B}^n \to \mathbf{B}^n$ satisfies $g(0) = 0$, $g(re_1) = te_1$ where $\rho(-s^\alpha e_1, s^\alpha e_1) = \rho(0, te_1)$ and hence $t = 2r^\alpha/((1+r')^\alpha + (1-r')^\alpha)$ by [Vu1, (2.17)]. The proof for g is complete. For the map h the proof is similar except that we use the K-quasiconformal mapping $m : x \mapsto |x|^{\beta-1}x,\ \beta = 1/\alpha$. Note that $m = f^{-1}$ and $t = 1/\mathrm{ch}(\alpha\ \mathrm{arch}(1/r))$. For the proof of (c) we apply (a), (b) together with [LV, (3.4), p.64]. $\square$

4.12. **Lemma.** *For $K > 1$, $c(K) \geq \log(\mathrm{ch}(K\mathrm{arch}(e))) \geq u(K-1)+1$, where $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$.*

Proof. From Lemma 4.11(c), we know that

$$\begin{aligned}\varphi_{1/K}(1/e) &\leq \frac{2/e^K}{(1+\sqrt{1-1/e^2})^K+(1-\sqrt{1-1/e^2})^K}\\ &= \frac{2}{(e+\sqrt{e^2-1})^K+(e-\sqrt{e^2-1})^K},\end{aligned}$$

hence

$$\begin{aligned}c(K) &= -\log\varphi_{1/K}(1/e)\geq -\log\left(\frac{2}{(e+\sqrt{e^2-1})^K+(e-\sqrt{e^2-1})^K}\right)\\ &= \log\left(\frac{(e+\sqrt{e^2-1})^K+(e-\sqrt{e^2-1})^K}{2}\right)\\ &= \log(\mathrm{ch}(K\mathrm{arch}(e)))\geq u(K-1)+1,\end{aligned}$$

where the last inequality follows easily from the mean value theorem, applied to the function $p(K)=\log(\mathrm{ch}(K\mathrm{arch}(e)))$. □

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